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NOTHING IS PART OF EVERYTHING

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NOTHING IS PART OF EVERYTHING[1]

Paolo Valore[2]

«Lui in sé non vale altro che nulla, e tutti i nulli
dell'universo sono eguali ad un sol nulla
in quanto alla loro sostanza e valore»
(Leonardo da Vinci, "Trattato della pittura")

1.Definitions

I will assume the language of general *Extensional* mereology, that is, a mereology without intensional operators (such as modal or temporal functions)[3]. The axiom system for mereology requires a single primitive concept: the parthood relation, symbolized as $\lhd$. The $\lhd$-relation[4] is assumed to be (a) *reflexive*, (b) *antisymmetric* and (c) *transitive*[5]. Note that the parthood relation is a partial *ordering*, as defined in set theory[6]. Since parthood is analogous to inclusion, I can define a relation somehow analogous to proper

[1] This article has been written during a semester at the Department of Philosophy of NYU, New York, as a Fulbright Visiting Scholar. I want to thank my students at the University of Milan for the discussions on section 6. A former version of some ideas of this paper has been presented in Valore (2008).
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[3] Extensionality essentially amounts to the assumption that two entities are identical if and only if they have the same parts. This means I will give up any distinction between entities made up of the same parts in different order (for any meaning of "order", both spatial or temporal). This assumption mirrors the fundamental principle of set theory: two sets are identical if and only if they have the same members. This choice does raise some problems for the intuitive notion of "part", but it is not relevant for the issue I want to discuss here.
[4] Cfr. Leśniewski (1992), pp. 131-132. Note that Leśniewski's notion of part is what I shall hereafter call "proper part", while what I define "part" is called, in his writings, "ingredient".
[5] For (a) cfr. Leśniewski (1992), § 2, theorem II. For (b) cfr. Leśniewski (1992), § 1, axiom I, relative to "proper parts", alongside definition I in § 2; cfr. also Simons (1987), p. 10. For (c) cfr. Leśniewski (1992), § 3, theorem IV; Tarski (1929), postulate I. Cfr. also Simons (1987), p. 10.
[6] This relation is in many respects analogous to inclusion among sets (this is obviously no more than an analogy, for mereology doesn't deal with sets of elements, but with entities and parts thereof). Consider for example the powerset $\mathscr{P}(U)$ of any set U: assuming "X_i" as variables for sets, the relation $\mathscr{R}$ defined as:

$$\forall X_1, X_2 \in \mathscr{P}(U) \; : \mathscr{R}(X_1, X_2) \iff X_1 \subseteq X_2$$

is a partial ordering based on inclusion between subsets.

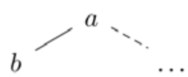
inclusion. Such a relation exists, and is called "proper parthood" ("$\lhd\lhd$"[7]). It can be considered as a special instance of $\lhd$, when it holds of (x,y) but not of (y,x) (in all cases, that is, when $x \neq y$)[8]:

$$(1) \qquad x \lhd\lhd y \equiv_{def} x \lhd y \land \neg(y \lhd x)$$

. The ordering introduced by "$\lhd\lhd$" can be rendered by a diagram as follows: all elements are represented by nodes; the antisymmetric relation

$$\big(\mathscr{R}(b,a) \land \neg\mathscr{R}(a,b)\big)$$

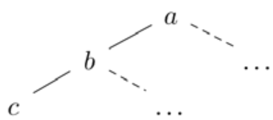
can be rendered by locating node *b* underneath the node *a*:



; the transitive relation

$$\Big(\Big(\mathscr{R}(b,a) \land \mathscr{R}(c,b)\Big) \rightarrow \mathscr{R}(c,a)\Big)$$

can be rendered by placing *b* underneath *a* and *c* underneath both *a* and *b*:



. Clearly, interpreting each node in this diagram as representing an entity, it could be read as a representation of the relations holding between entity *a* and its proper parts. If *a* has

[7] The double symbol used here reproduces a use found in in Leonard & Goodman (1940) and Simons (1987), p. 10.

[8] Cfr. Leśniewski (1992), § 1, theorem I; Tarski (1929), definition I. Cfr. also Simons (1987), p. 10. Note that an easier way of stating the definition might have been:

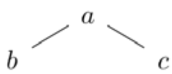
$$x \lhd\lhd y \equiv_{def} x \lhd y \land x \neq y.$$

. I have nonetheless preferred to define proper parthood only in terms of the parthood relation. The axiom of antisimmetricy

$$\forall x \forall y \left(x < y \land y < x \rightarrow (x = y)\right)$$

could still be interpreted as a definition of =, thereby making identity a concept derived from the primitive notion of part: two entities x and y are identical if and only if $x \lhd y$ and $y \lhd x$.

at least two proper parts (*b* and *c*) which aren't part of each other, I can represent the relation holding between them as follows:



. From the primitive concept of part, and its axioms, it is possibile to define the notion of *atom* as a derived concept (which should occupy the lowermost position in the diagrams). I'll define "atom" (referred to with indexed variables such as $\mathfrak{A}_1$, $\mathfrak{A}_2$, …)[9] as an entity with no proper parts (that is, entity *x* such that, if entity *y* is part of it, then it is identical to it)[10]:

$$(2) \qquad \mathfrak{A}_i = x \ \equiv_{def} \ \forall y \, (y \lhd x \rightarrow y = x)$$

. This definition has the advantage of being consistent with the intuitive idea of an atom as the residue of an object's progressive decomposition[11]. From the primitive concept of part, "overlapping"[12] can also be introduced: intuitively, two entities overlap if they share some content. I will indicate that *x* overlaps with *y* as follows: "$x \circ y$":

$$(3) \qquad x \circ y \equiv_{def} \exists z (z \lhd x \wedge z \lhd y)$$

. This can be rendered as follows:

[9] For the ease of the reader I reserve a set of dedicated variables to atoms, although strictly speaking a predicate with the same definition would have been more perspicuous.

[10] Meixner (1997), p. 29 offers an equivalent definition:

$$\neg \exists y (y \lhd x \wedge x \neq y)$$

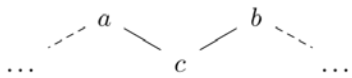
– notational conventions are uniformed to the standards I adopted here.

[11] The admittance of atoms, however, is not inevitable. One might design a mereological system rejecting the notion of atom, adding an axiom as

$$\forall x \exists y \, (y \lhd \lhd x)$$

and generating diagrams with an upper bound (*U*) but no lower bound. Such a universe is made of entities with proper parts, which have proper parts themselves, and so on. This possibility, since $\lhd\lhd$ is anti-symmetric, is granted only by an infinite cardinality of models for the theory – cfr. Varzi (1996). Such a model could be provided by open sets in a euclidean space, interpreting parthood as set-theoretic inclusion – cfr. Tarski (1935).

[12] Cfr. Goodman (1958) and Simons (1987), pp. 11-12.



. Overlapping is reflexive, as is parthood, but, unlike the latter, it is symmetric and intransitive[13]. Two objects that don't overlap are said to be *discrete*[14]. To state that *x* and *y* are discrete I write as follows: "$x \oslash y$".

$$(4) \qquad x \oslash y \equiv_{def} \neg(x \circ y)$$

. Or, which is analogous – since parthood and overlapping are mutually definable – two objects are discrete if they share no part[15]. All objects are thus either atoms or fusions of atoms, where "fusion" is taken to informally[16] mean the entity consisting of certain entities as its only parts[17]. If I apply without restrictions the idea of fusion, I can generate the notion of a "universe", a total mereological fusion "*U*". This plays a similar role to the universal class in set theory, with a significant difference: *U* is an individual composed of all parts of all individuals, and not a set thereof.

2. Trying to Fill the Gap

The mereological analog of the empty set should be an individual, which might be called "null object"[18]. Mereology, in its traditional formulation, avoids all reference to null objects[19] (as we will see, with good reasons). Some systems, however, do try to define a

[13] Naturally, if it is the case that exactly the same individuals overlap with two objects, then the two objects are identical:

$$\forall z \Big((z \circ x \leftrightarrow z \circ y) \rightarrow x = y \Big)$$

[14] Cfr. Tarski (1929), def. II, p. 25.
[15] Cfr. Rescher (1955), def. I, p. 9:

$$(8) \qquad x \oslash y \equiv_{def} \neg \exists z (z \lhd x \wedge z \lhd y)$$

[16] More formally, I can define the fusion of objects *x* and *y* (in symbols: "$x \oplus y$") as follows:

$$x \oplus y \equiv_{def} \iota z \forall w \Big(w \circ z \leftrightarrow (w \circ x \vee w \circ y) \Big)$$

that is: the fusion of two individuals is the individual overlapping exactly with all individuals overlapping with either of the two.
[17] I will not discuss here the difference between *sum* and *fusion*, for it is not relevant for my purpose. On such distinction, cfr. Simons (1987).
[18] Cfr. Martin (1943), p. 3; Carnap (1947), p. 65.
[19] Cfr. Leonard & Goodman (1940), p. 46, and Casati & Varzi (1999).

null object $\mathfrak{N}$ as a particular instance of atom[20]. The admission of a null object could seem ontologically suspicious, but it has the advantage of allowing mereology to develop the analogy with set theory to its fullest extent[21]. From a purely formal point of view, such an introduction raises no problems; from an ontological point of view, on the other hand, one might wonder what sort of relation holds between atoms and the null object. Actually, it is the ontological point of view which has prompted us to test whether mereology could be a good replacement of set theory[22].

Since it is true of the empty set that:

$$\forall X\ (\emptyset \subseteq X)$$

, I could attempt an initial definition of "$\mathfrak{N}$", adding to the requirements imposed on atoms an analog of such condition:[23]:

$$(6) \qquad \mathfrak{N} \equiv_{def} \iota x \forall y \big((y \lhd x \rightarrow x = y) \wedge \forall z\ (x \lhd z) \big)$$

. Clearly, the introduction of $\mathfrak{N}$ shouldn't imply the non-existence of any atom other than $\mathfrak{N}$, for it has been defined as a special instance of $\mathfrak{A}_i$, through an additional specification. On the other hand, if I accept the aforementioned definitions, atoms and null object collapse onto each other. This is even more significant if I consider that the null object is an individual constant, while atom is a predicate logically capable of being true of more than one individual. Condition $\forall z(x \lhd z)$ (alongside the definition of $\lhd$) implies that one of the following must hold: either (a) $x = z$ or (b) $x \lhd\lhd z$.

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[20] Bunge (1966) proposes even more than one kind of null individual. A particular kind of atom has the same role in Meixner (1997), p. 29: it is the "minimal atom", which is part of every object.

[21] If mereology is to represent a serviceable replacement of set theory, it must try to rebuild its fundamental notions through the tools it has (that is, nothing but individuals and the primitive parthood relation).

[22] Goodman, for instance, uses it to avoid all reference to entities such as sets, limiting himself to a calculus of individuals. The stance at ontological neutrality was first implied, at the discipline's earliest outsets (such as Leśniewski's) in the interpretation of existential quantifiers. Classical mereology, to avoid suspicious ontological commitments, imposed a *non-referential quantification*. In quantification theory an expression such as

$$P(a) \vdash \exists x\ (P(x))$$

isn't valid, for I can't be sure that there is something which is *P*. Early mereology, by imposing a non-referential quantification, always verified the aforementioned inference, limiting the distinction between quantifiers to a distinction between a universal and particular distribution of predicate *P*. In Leśniewski's ontology, for instance, it is provable that for some *a*, *a* doesn't exist (cfr. Simons (1987), p. 20). On the other hand, it is provable that mereology's alleged ontological innocence is more apparent than real: cfr. Carrara & Martino (2001), pp. 75-110, and Yi (1999).

[23] Cfr. Meixner (1997), pp. 29-31.

(a) This clearly can't be the case, for the definition is nontrivial only if $z \neq \mathfrak{N}$. If $\mathfrak{N} \lhd z \wedge \neg \mathfrak{N} \lhd \lhd z$, then $x \lhd z$ becomes a trivial consequence of the reflexivity of $\lhd$: the *x* in the previous definition is taken for $\mathfrak{N}$, so that $x \lhd z$ would amount, in this case, to $\mathfrak{N} \lhd \mathfrak{N}$, which is an instance of $\forall x \ (x \lhd x)$.

(b) Suppose then that $\mathfrak{N} \lhd \lhd z$. Applying the definition of "$\lhd \lhd$" I obtain:

$$\forall z \left(z \neq \mathfrak{N} \rightarrow (\mathfrak{N} \lhd z \wedge \neg z \lhd \mathfrak{N}) \right)$$

Which means: every entity which is not $\mathfrak{N}$ has $\mathfrak{N}$ as its proper part (while $\mathfrak{N}$ has no proper parts of its own).

$\mathfrak{N}$ would then be *a proper part of every other entity* in the universe. This means that the last residue of the decomposition of any fusion will be $\mathfrak{N}$, and "atom" becomes a useless synonym of "null object": for every entity which is not $\mathfrak{N}$ has at least a proper part, and atoms have no proper parts, then $\mathfrak{N}$ is the only atom. It is clear, on the other hand, that I can't give up characterizing $\mathfrak{N}$ as an atom: the null object can't have proper parts, for there cannot be a part of nothing which is different from nothing.

An alternative hypothesis could suppose *not* to consider $\mathfrak{N}$ a particular instance of $\mathfrak{A}_i$, instead defining the latter in terms of $\mathfrak{N}$. The idea is to define the null object as follows:

$$\text{(6*)} \qquad \mathfrak{N} \equiv_{def} \iota x \forall y \ (x \lhd y)$$

and then:

$$\text{(2*)} \qquad \mathfrak{A}_i = x \equiv_{def} \forall y \ (y \lhd \lhd x \rightarrow y = \mathfrak{N})$$

. In other terms, an atom is an object *x* such that

$$\neg \exists y \ (y \lhd \lhd x \wedge y \neq \mathfrak{N})$$

. This amounts to saying that the only proper part of atoms is the null object. In this case, obviously, the definition can't hold for $\mathfrak{N}$ as the *x* in $y \lhd \lhd x$, for no object can be a proper part of itself; the definition of $\mathfrak{A}_i$ would nonetheless be satisfied by $\mathfrak{N}$ as well, because if $x = y = \mathfrak{N}$, then the antecedent in the definition would be false (although the consequent would be true), so as to satisfy the whole formula. The atom would be defined in a somewhat counter-intuitive way, but it would be the only one compatible with the assumption that the null object is part of everything. This would make $\mathfrak{N}$ an instance of $\mathfrak{A}_i$ based on the truth table for →, in particular: "false → true".

Nothing prevents our system from avoiding such definition and subsequently not defining $\mathfrak{R}$ as an atom, which wouldn't be in itself a contradiction, for I can always give up the intuitive concept of atom within a context of prescriptive ontology. An atom would then not be a simple part, but a simple part different from $\mathfrak{R}$:

$$(2^{**}) \qquad \mathfrak{A}_i = x \equiv_{def} \exists y (y \lhd \lhd x) \land \forall y \, (y \lhd \lhd x \rightarrow y = \mathfrak{R})$$

In this way, neither the <false→true> nor the <false→false> cases would be applicable. If the antecedent is false, I can't verify the first clause in the conjunction, but if the antecedent is true, then necessarily $y = \mathfrak{R}$. If I accept this revised definition I must reject what I just said, that $\mathfrak{R}$ is a particular instance of $\mathfrak{A}_i$. At the beginning, $\mathfrak{R}$ was defined as such, by adding a clause to the definition of atom:

$$\mathfrak{A}_i = x \ \equiv_{def} \ \forall y \, (y \lhd x \rightarrow y = x)$$

so that:

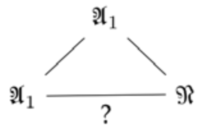
$$\mathfrak{R} \equiv_{def} \iota x \forall y \big((y \lhd x \rightarrow x = y) \land \forall z \, (x \lhd z) \big)$$

. I can naturally keep this definition, which is now independent from (2**).

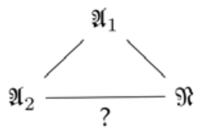
5. Enigmatic Trees

By accepting the second definition of atom, what I obtain is that an atom is every entity which has no proper parts besides the null object. I could now represent the relation between an atom $\mathfrak{A}_1$ and $\mathfrak{R}$ as follows:

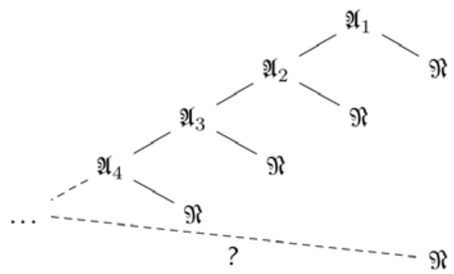
. This diagram, however, clearly shows that something is wrong, for the atom cannot have a single proper part. If I want to state that $\mathfrak{A}_1 \neq \mathfrak{R}$, the axiom of extensionality requires $\mathfrak{A}_1$ to be a fusion of $\mathfrak{R}$ and at least another proper part different from $\mathfrak{R}$. If it were a fusion of $\mathfrak{N}$ and $\mathfrak{N}$, then $\mathfrak{N}$ wouldn't be a proper part of it, so that, once again, $\mathfrak{A}_1 = \mathfrak{N}$. To save the notion of "atom", then, I could specify that atoms ought to be the fusions of $\mathfrak{N}$ and the atom itself. But this relation couldn't be rendered in a meaningful diagram:



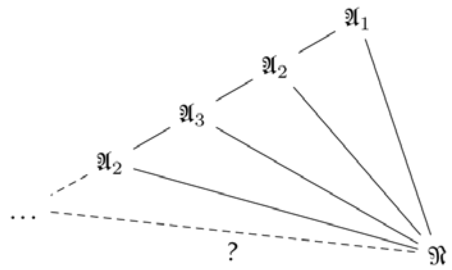
. This can't be accepted, for $\mathfrak{A}_1$ can't be a proper part of itself. I could perhaps rewrite it thus:



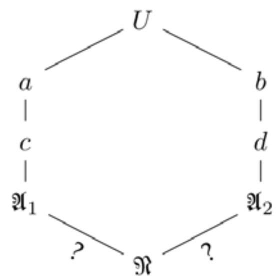
where $\mathfrak{A}_1 \neq \mathfrak{A}_2$. And clearly if this were the case, the true atom would be $\mathfrak{A}_2$ – but this would once again be meaningless, for, since there are no proper parts of $\mathfrak{A}_1$ besides $\mathfrak{A}_2$ ($\mathfrak{N}$ wouldn't count, for it is part of both), we'd have one more time $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{N}$. Anyway, even if this diagram made any sense (and it doesn't), it would generate another cascade of atoms: $\mathfrak{A}_2$ should have $\mathfrak{N}$ as its proper part, and as such it should have a proper part besides $\mathfrak{N}$ – suppose, $\mathfrak{A}_3$, and so on:



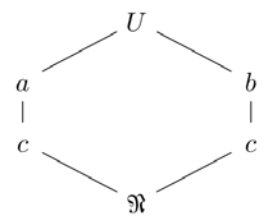
. This is absurd, because $\mathfrak{N}$ is a single object (this was known to Leonardo da Vinci: “all nothings in the universe are just one single nothing in their substance”): *there is just one null object*.



In general, I could wonder: if I imagine two objects *a* and *b*, such that $a \oslash b$, which are fusions, respectively, of *c* and $\mathfrak{A}_1$ and of *d* and $\mathfrak{A}_2$, so that $\mathfrak{A}_1 \triangleleft c$ and $\mathfrak{A}_2 \triangleleft d$:

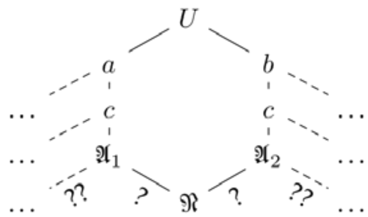


, what is the relationship between the two atoms $\mathfrak{A}_1$, $\mathfrak{A}_2$ and the null object? Can they have as *sole* proper part the same null object? In this case, they would share a proper part, and so $\mathfrak{A}_1 \circ \mathfrak{A}_2$, but this would mean $a \circ d$, which contradicts our initial hypothesis. Moreover, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have the same proper parts (nothing but $\mathfrak{N}$), so, in an Extensional mereology, they should be the same entity, and the diagram should be rewritten as:



and so on, through $c = e$ and up to $a = d$, from our initial assertion that $a \oslash d$! Note, moreover, that this unexpected result would render $a = d = U$, and so $U = \mathfrak{N}$, which would likely mean something similar to everything being identical with nothing.

The problem for *a*, *b*, *c* and *d* can be trivially solved by considering that, if *ex hypothesi* $a \oslash b$, then $a \neq b$, and so they must have distinct proper parts, and hence they must have more than one. The same argument should in turn apply to *c* with $\mathfrak{A}_1$ and *d* with $\mathfrak{A}_2$: if the atoms are *proper* parts of *c* and *d*, respectively, they can't be identical to them. So they can't be their only proper parts:



. This would mean that, if an object has a proper part, then it must have at least two (that is, another proper part distinct from the first). This request is made by means of a supplementation axiom, which imposes that a fusion always be a fusion of $n > 1$ parts.

If any supplementation axiom adds a clause to *c* being proper part of *a* and to $\mathfrak{A}_1$ being proper part of *c* (and the same for the other branch of the diagram), I won't be able to satisfy that clause by adding a ramification at the lower end of the diagram.

Our definition of atom implies it has no proper parts but $\mathfrak{N}$.

When I try to apply the supplementation axiom to the definition of atoms and the null object, the only solution would be considering atoms as fusions of themselves and the null object – which would end up in cascading diagrams as the ones seen before. The only solution would be stating that the supplementation axiom and the principle of extensionality do not apply to atoms and to the null object.

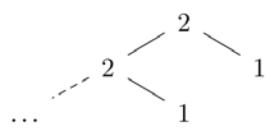
All this argument, in the end, seems to prove how the acceptance of a null object in a mereological system will complicate the latter up to the point of absurdity or inconsistency (or to adding whole sets of *ad hoc* axioms).

6. A Model

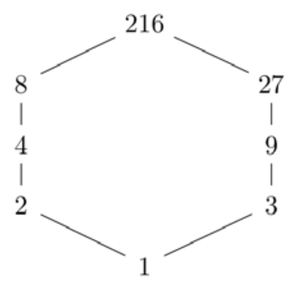
Suppose now to assume the set ℕ of the natural numbers as domain of quantification and to read the ◁-relation as "is a divisor of".

Consequently, proper parthood has to be read as "is a divisor of but not identical to" and overlapping as "having a common divisor different than 1". In particular, an atom is now any *prime* number (1 included or excluded, according to the decision to accept the first definition or the second one).

Please note that the definition of identity as "having exactly the same parts" still holds within this interpretation: two natural numbers are the same natural number if and only if the have the same prime divisors. Going back to the problematic diagrams I built before, I can now remark that an atom (a prime number such as 2) has, as its parts, the atom itself and the null object (1). So that



now makes complete sense. So does the case of the decomposition of a number in factors that do not have common divisors different than 1:



. The price for this interpretation is that we cannot have any "Universe" in this model, that is, that in order to accept both atoms and the null-object, we seem forced to design a mereological system so to generate diagrams with a lower bound but no upper bound.

And the fact that we cannot generate the total mereological fusion makes the analogy with set theory (where it is impossible to generate the "Universal set") yet more intriguing.

Bibliography

BUNGE, M. 1966. “On Null Individuals”, *Journal of Philosophy*, 63, pp. 776–778.

CARNAP, R. 1947. *Meaning and Necessity: A Study in Semantics and Modal Logic*, The University of Chicago Press, Chicago.

CARRARA, M., & MARTINO, E. 2001. “On the Alleged Innocence of Mereology”, *The Review of Symbolic Logic*, 2, pp. 164–174.

CASATI, R., & VARZI, A. 1999. *Parts and Places: The Structures of Spatial Representation*, The MIT Press, Cambridge.

GOODMAN, N. 1958. “On Relations that Generate”, *Philosophical Studies*, pp. 65–66.

LEONARD, H. S., & GOODMAN, N. 1940. “The Calculus of Individuals and its Uses”, *Journal of Symbolic Logic*, pp. 45–55.

LEŚ NIEWSKI, S. 1992. “Foundations of the General Theory of Sets”, in SURMA, S. J., SRZEDNICKI, J., & BARNETT, D. I. (eds), Collected Works, pp. 129-173, Kluwer, Dordrecht-Boston-London.

MARTIN, R. M. 1943. “A Homogeneous System for Formal Logic”, *Journal of Symbolic Logic*, pp. 1–23.

MEIXNER, U. 1997. *Axiomatic Formal Ontology*, Kluwer, Dordrecht-Boston- London.

RESCHER, N. 1955. “Axioms for the Part Relation", *Philosophical Studies*, pp. 8-11.

SIMONS, P. 1987. *Parts. A Study in Ontology*, Clarendon Press, Oxford.

TARSKI, A. 1929. “Les fondements de la géométrie des corps”, *Annales de la societé de mathématiques polonaise*, pp. 29–33.

TARSKI, A. 1935. “Zur Grundlegung der Booleschen Algebra. I”, *Fundamenta mathematicae*, pp. 177–198.

VARZI, A. 1996. “Parts, Wholes, and Part-Whole Relations: The Prospects of Mereotopology”, *Data and Knowledge Engineering*, 20, pp. 259–286.

VALORE, P. 2008. *L’inventario del mondo. Guida allo studio dell’ontologia*, UTET, Torino.

YI, B.U. 1999. “Is Mereology Ontologically Innocent?”, *Philosophical Studies*, pp. 141–160.